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B.Sc Part III

MOORE OSGOOD THEOREM

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Theorem Let the Simultaneous lim $\underset{\substack{x \to a \\ y \to b}}{Lt} f(x,y)$ exist and be equal to l and let the limit $\underset{x \to a}{Lt} f(x,y)$ exists for each constant value of y in the nhd of $y = b$ and likewise let the limit $\underset{y \to b}{Lt} f(x,y)$ exist for ~~each~~ each constant value of x in the nhd of $x = a$

Then $\underset{x \to a}{Lt} \ \underset{y \to b}{Lt} f(x,y) = \underset{y \to b}{Lt} \ \underset{x \to a}{Lt} f(x,y)$

Proof Since the limit $\underset{x \to a}{Lt} f(x,y)$ exist for each value of y in the nhd of $y = b$, we shall obtain an aggregate of these limiting values which defines a function of y say $F(a,y)$

Thus we have $\underset{x \to a}{Lt} f(x,y) = F(a,y)$

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where $F(a,y)$ may or may not be identical with $f(a,y)$

Let $\epsilon > 0$ be given

Since $\lim_{x \to a} f(x,y) = F(a,y)$, therefore there exists $\delta_1 > 0$ such that for each value of y in the nhd of $y = b$ i.e. for $|y-b| < \delta$, we have

$$|F(a,y) - f(x,y)| < \frac{\epsilon}{2} \quad \text{--- (2)}$$

For all x satisfying $|x-a| < \delta_2$

Also from the existence of simultaneous limit at (a,b), there exists $\delta_2 > 0$ such that

$$|f(x,y) - l| < \frac{\epsilon}{2} \quad \text{--- (3)}$$

For all x, y satisfying $|x-a| < \delta_2$
$|y-b| < \delta_2$

Let $\delta = \min(\delta_1, \delta_2)$

Then we have $|F(a,y) - l|$

$= |F(a,y) - f(x,y) + f(x,y) - l|$

$< |F(a,y) - f(x,y)| + |f(x,y) - l|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$; by virtue of ② & ③

It follows, therefore that

$$\underset{y \to b}{Lt}\ F(a,y) = l$$

$$\Rightarrow \underset{y \to b}{Lt}\ \underset{x \to a}{Lt}\ f(x,y) = l; \text{ by } ①$$

Similarly it can be shown that

$$\underset{x \to a}{Lt}\ \underset{y \to b}{Lt}\ f(x,y) = l$$

Thus $\underset{\substack{x \to a \\ y \to b}}{Lt} f(x,y) = \underset{x \to a}{Lt}\ \underset{y \to b}{Lt}\ f(x,y)$

$$= \underset{y \to b}{Lt}\ \underset{x \to a}{Lt}\ f(x,y) = l.$$

Proved